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Abstract

We define Letac-Wesolowski-Matsumoto-Yor (LWMY) functions as decreasing functions from $(0,\infty)$ onto $(0,\infty)$ with the following property: there exist independent, positive random variables X and Y such that the variables $f(X+Y)$ and $f(X)-f(X+Y)$ are independent. We prove that, under additional assumptions, there are essentially four such functions. The first one is $f(x)=1/x$. In this case, referred to in the literature as *the Matsumoto-Yor property*, the law of X is generalized inverse Gaussian while Y is gamma-distributed. In the three other cases, the associated densities are provided. As a consequence, we obtain a new relation of convolution involving gamma distributions and Kummer distributions of type 2.

Keywords: Gamma distribution; generalized inverse Gaussian distribution; Matsumoto-Yor property; Kummer distribution.

Introduction

A lot of papers have been devoted to the generalized inverse Gaussian (GIG) distributions since their definition by Good (1953)(see for instance Barndorff-Nielsen and Halgreen (1977), Letac and Seshadri (1983), Vallois (1989), Vallois (1991)).

The GIG distribution with parameters $\mu \in \mathbb{R}$, $a, b > 0$ is the probability measure :

$$GIG(\mu, a, b)(dx) = \left(\frac{b}{a}\right)^{\mu} \frac{x^{\mu-1}}{2K_\mu(ab)} e^{-\frac{1}{2}(a^2x^{-1}+b^2x)} \mathbf{1}_{(0,\infty)}(x)dx \tag{0.1}$$

where K_μ is the classical McDonald special function.

GIG distributions can also be defined as probability measures on the set of positive definite matrices (Letac and Wesolowski, 2000). Here, we only deal with one-dimensional GIG distributions.

1) Let us stress the close links between GIG, gamma distributions and the function $f_0(x) = 1/x$ $(x > 0)$.

a) The family of GIG distributions is invariant under f_0: we can easily deduce from (0.1) that the image of GIG(μ, a, b) by f_0 is GIG$(-\mu, b, a)$.

b) Barndorff-Nielsen and Halgreen (1977) proved:

$$GIG(-\mu, a, b) * \gamma(\mu, \frac{b^2}{2}) = GIG(\mu, a, b), \quad \mu,\ a,\ b > 0 \tag{0.2}$$

where $\gamma(\mu, b^2/2)(dx) = \frac{b^{2\mu}}{2^\mu \Gamma(\mu)} x^{\mu-1} \exp -\frac{b^2}{2} x \, \mathbf{1}_{(0,\infty)}(x) dx$.
Therefore if $X > 0$ and $Y \sim \gamma(\lambda, a^2/2)$ are independent r.v.'s and $X \sim \mathrm{GIG}(-\lambda, a, a)$ then

$$X \stackrel{(d)}{=} f_0(X+Y). \tag{0.3}$$

Letac and Seshadri (1983) proved that (0.3) characterizes GIG distributions of the type $\mathrm{GIG}(-\lambda, a, a)$. Namely, if X and Y are positive r.v.'s such that $Y \sim \gamma(\lambda, a^2/2)$ and (0.3) holds then $X \sim \mathrm{GIG}(-\lambda, a, a)$. The authors have also obtained a similar characterization of $\mathrm{GIG}(-\lambda, a, b)$ but the formulation is more complicated.

c) Since relation (0.2) is only formulated in terms of probability measures, we can ask whenever it admits an almost sure realization. That consists in giving a triplet of r.v.'s X, Y, Z such that $Z = X + Y$, X and Y are independent, $X \sim \mathrm{GIG}(-\mu, a, b)$ and $Y \sim \gamma(\mu, b^2/2)$. Almost sure realizations have been given by Bhattacharya and Waymire (1990) in the case $\mu = \frac{1}{2}$ and Vallois (1991) for any $\mu > 0$. The authors have considered a family of transient diffusions $(D^\nu(t),\, t \geq 0)$ on $[0, \infty)$ which depends on a parameter ν. It has been proved that there exist $x_0 > 0$ and ν_0 such that $D^{\nu_0}(0) = x_0$ and

$$T_0 := \inf\{t \geq 0, D^{\nu_0}(t) = 0\} \sim \mathrm{GIG}(-\mu, a, b)$$

$$L_0 := \sup\{t \geq 0, D^{\nu_0}(t + T_0) = 0\} \sim \gamma(\mu, b^2/2).$$

The strong Markov property at time T_0 implies that $X = T_0$ and $Y = L_0$ are independent and $Z = X + Y = \sup\{t \geq 0, D^{\nu_0}(t) = 0\}$.
In Madan, Roynette and Yor (2008), the Black-Scholes formula in finance is expressed in terms of the distribution function of GIG variables (see Equations (25), (29), (30) in this reference).

Let $(\xi_1(t),\, t \geq 0)$ and $(\xi_2(t),\, t \geq 0)$ be two Lévy processes such that $\xi_1(t) \sim \gamma(t, \beta_1^2/2)$ and $\xi_2(t) \sim \gamma(t, \beta_2^2/2)$ for any $t > 0$, where $\beta_1, \beta_2 > 0$. Recall that for $i = 1, 2$ the process $(\xi_i(t))$ is a subordinator and consequently $t \mapsto \xi_i(t)$ is non decreasing. Let us define the random time

$$\begin{aligned} N &= \inf\{n \geq 0;\ \xi_1(\alpha_1 + n - 1)\xi_2(\alpha_2 + n - 1) \leq 1 < \xi_1(\alpha_1 + n)\xi_2(\alpha_2 + n - 1)\} \\ &= \inf\{n \geq 0;\ \xi_1(\alpha_1 + n - 1) \leq f_0(\xi_2(\alpha_2 + n - 1)) < \xi_1(\alpha_1 + n)\} \end{aligned} \tag{0.4}$$

where $\alpha_1, \alpha_2 > 0$ and by convention $\inf \emptyset = +\infty$.

Then, it has been proved in Vallois (1989, theorem on p.446) that, conditionally on $\{N < \infty\}$, the r.v.'s

$$X := \xi_2(\alpha_2 + N - 1), \quad Y := \xi_2(\alpha_1 + N - 1) - \xi_2(\alpha_2 + N - 1) \tag{0.5}$$

are independent and their laws are $\mathrm{GIG}(-\mu, \beta_1, \beta_2)$ and respectively $\gamma(\mu, \beta_2^2/2)$, where $\mu = \alpha_1 - \alpha_2$. This yields a second almost sure realization of (0.2). Furthermore, the r.v.'s

$$U := \frac{1}{X+Y} = \xi_2(\alpha_1+N-1), \;\; V := \frac{1}{X} - \frac{1}{X+Y} = \frac{1}{\xi_2(\alpha_2+N-1)} - \frac{1}{\xi_2(\alpha_1+N-1)}$$

are independent.

2) We now focus on *the Matsumoto-Yor property* involving the GIG and gamma distributions on the one hand and the function f_0 on the second hand . Let X and Y be two independent r.v.'s such that

$$X \sim \mathrm{GIG}(-\mu, a, b), \;\; Y \sim \gamma(\mu, b^2/2), \quad (\mu, a, b > 0). \tag{0.6}$$

Then

$$U := \frac{1}{X+Y} = f_0(X+Y), \;\; V := \frac{1}{X} - \frac{1}{X+Y} = f_0(X) - f_0(X+Y) \tag{0.7}$$

are independent and

$$U \sim \mathrm{GIG}(-\mu, b, a), \;\; V \sim \gamma(\mu, a^2/2). \tag{0.8}$$

The case $a = b$ was proved by Matsumoto and Yor (2001) and a nice interpretation of this property via Brownian motion was given by Matsumoto and Yor (2003). The case $\mu = -\frac{1}{2}$ of the Matsumoto-Yor property can be retrieved from an independence property established by Barndorff-Nielsen and Koudou (1998) (see Koudou, 2006).

Note that the r.v.'s X and Y introduced in (0.5) satisfy the Matsumoto-Yor property. Letac and Wesolowski (2000) proved that the Matsumoto-Yor property holds for any $\mu, a, b > 0$ and characterizes the GIG distributions. More precisely, consider two independent and non-Dirac positive r.v.'s X and Y such that U and V defined by (0.7) are independent, then there exist $\mu, a, b > 0$ such that (0.6) holds. Obviously this property is similar to the one given by Letac and Seshadri (1983).
Massam and Wesolowski (2004) derived a tree-version of the Matsumoto-Yor property.

3) The origin of this paper is to understand the link between the function $f_0 : x \mapsto 1/x$ and the GIG distributions in the Matsumoto-Yor property. It is convenient to introduce the following transformation T_f associated with a decreasing function $f : (0, \infty) \to (0, \infty)$:

$$\begin{aligned} T_f \,:\, (0,\infty)^2 &\to (0,\infty)^2 \\ (x, y) &\mapsto (f(x+y), f(x) - f(x+y)). \end{aligned}$$

Suppose that f is one-to-one, then T_f is bijective and $(T_f)^{-1} = T_{f^{-1}}$. Let $(U, V) = T_f(X, Y)$ where X and Y are positive r.v.'s, namely

$$U = f(X+Y), \; V = f(X) - f(X+Y). \tag{0.9}$$

We can recover X and Y from (U, V):

$$X = f^{-1}(U+V), \; Y = f^{-1}(U) - f^{-1}(U+V). \tag{0.10}$$

Considering $f = f_0$ and independent r.v.'s X and Y, the Matsumoto-Yor property asserts that the r.v.'s $f_0(X+Y)$ and $f_0(X) - f_0(X+Y)$ are independent as soon as $X \sim \mathrm{GIG}(-\mu, a, b)$ and $Y \sim \gamma(\mu, b^2/2)$ for some $\mu, a, b > 0$.

Obviously, the Matsumoto-Yor property can be reexpressed as follows: the image of the probability measure (on $\mathbb{R}_+^2$) $\mathrm{GIG}(-\mu, a, b) \otimes \gamma(\mu, b^2/2)$ by the transformation T_{f_0} is the probability measure $\mathrm{GIG}(-\mu, b, a) \otimes \gamma(\mu, a^2/2)$. This formulation of the Matsumoto-Yor property joined with the Letac and Wesolowski result lead us to determine the triplets (μ_X, μ_Y, f) such that

1. μ_X, μ_Y are probability measures on $(0, \infty)$,
2. $f : (0, \infty) \to (0, \infty)$ is bijective and decreasing,
3. if X and Y are independent r.v.'s such that $X \sim \mu_X$ and $Y \sim \mu_Y$ then the r.v.'s U and V defined by (0.9) are independent.

Unfortunately we have not been able to solve this question without restriction. Our method can be applied supposing that f is smooth and μ_X and μ_Y have smooth density functions (see Theorem 2.1 for details). After long and sometimes tedious calculations we prove (cf Theorem 1.2) that there are only four classes $\mathcal{F}_1, \dots, \mathcal{F}_4$ of functions f such that T_f keeps the independence property. The first class $\mathcal{F}_1 = \{\alpha/x;\, \alpha > 0\}$ corresponds to $f = f_0$. In Appendix 6.1 we prove that the only possible distributions for X and Y are GIG and gamma respectively. Thus, we recover the result of Letac and Wesolowski under stronger assumptions. The proof of Letac and Wesolowski is completely different since the authors made use of Laplace transforms and a characterization of the GIG laws as the distribution of a continued fraction with gamma entries (a property close to (0.3)). We have not been able to develop a proof as elegant as theirs. The reason is that, with $f = f_0$ we have algebraic properties (for instance continued fractions), while these properties are lost if we start with a general function f.

Then, for any $f \in \mathcal{F}_i$, $1 \le i \le 4$ we have been able to give the corresponding distributions of X and Y and the related laws of U and V (see Theorem 1.4, Remark 1.5 and Theorem 1.14).

It is worth pointing out that one interesting feature of our analysis is an original characterization of the families of distributions $\{\beta_\alpha(a, b, c);\, a, b, \alpha > 0,, c \in \mathbb{R}\}$ and the Kummer distributions $\{K^{(2)}(a, b, c);\, a, c > 0,, b \in \mathbb{R}\}$ (see (1.18) and (1.33) respectively). The Kummer distributions appear as the law of some random continued fractions (see Marklov *et al*, 2008, p.3393 mentioning a work by Dyson (1953) in the setting of random matrices).

As by-products of our study we obtain new relations of convolution. For simplicity we only detail the case of Kummer distributions. The Kummer distributions of type 2 and the gamma distributions are closely connected since:

$$K^{(2)}(a, b, c) * \gamma(b, c) = K^{(2)}(a + b, -b, c). \tag{0.11}$$

Obviously, this relation is similar to (0.2).

Inspired by the result of Letac and Wesolowski (2000) and Theorem 1.6, we can ask if a characterization of Kummer distributions could be obtained (cf Remark 1.8).

As recalled in the above item c), there are various almost sure realizations of convolution identities involving GIG distributions (see (0.2) and the convolution coming from the Matsumoto-Yor property. One interesting open question derived from our study would be to determine a r.v. Z with distribution $K^{(2)}(a+b,-b,c)$ which can be decomposed as the sum of two explicit independent r.v.'s X and Y such that $X \sim K^{(2)}(a+b,-b,c)$ and $Y \sim \gamma(b,c)$.

The paper is organized as follows. We state our main results in Section 1. In Section 2 we give, under smoothness assumptions, a key differential equation involving f and the log densities of the independent r.v.'s X and Y such that $f(X+Y)$ and $f(X)-f(X+Y)$ are independent (cf Theorem 2.1). Based on this result we prove in Section 3 (cf Theorem 3.8) that there are only four classes $\mathcal{F}_1, \dots, \mathcal{F}_4$ of such functions f. The theorems stated in Section 2 are proved in Section 4. Some technical proofs have been postponed in Appendix 5.

1 Main results

Definition 1.1 *Let $f : (0,\infty) \to (0,\infty)$ be a decreasing and bijective function.*

1. *Associated with f let us consider the transformation*

$$\begin{aligned} T_f \,:\, (0,\infty)^2 &\to (0,\infty)^2 \\ (x,y) &\mapsto (f(x+y), f(x)-f(x+y)). \end{aligned} \tag{1.12}$$

The transformation T_f is one-to-one and if f^{-1} is the inverse of f, then

$$(T_f)^{-1} = T_{f^{-1}}. \tag{1.13}$$

2. *Let X and Y be two independent and positive random variables. Let us define*

$$(U,V) = T_f(X,Y) = (f(X+Y), f(X)-f(X+Y)). \tag{1.14}$$

f is said to be a LWMY function with respect to (X,Y) if the random variables U and V are independent.
f is said to be a LWMY function if it is a LWMY function with respect to some random vector (X,Y).

One aim of this paper is to characterize LWMY functions.

Let us introduce

$$f_1(x) = \frac{1}{e^x - 1}, \quad x > 0, \tag{1.15}$$

$$g_1(x) = f_1^{-1}(x) = \ln\left(\frac{1+x}{x}\right), \quad x > 0 \tag{1.16}$$

and, for $\delta > 0$,

$$f_\delta^*(x) = \log\left(\frac{e^x+\delta-1}{e^x-1}\right), \quad x > 0. \tag{1.17}$$

Theorem 1.2 *Let $f : (0,\infty) \to (0,\infty)$ be decreasing and bijective. Under some additional assumptions (see Theorem 2.1 and Equation (3.3)), f is a LWMY function if and only if, either $f(x) = \frac{\alpha}{x}$ or $f(x) = \frac{1}{\alpha} f_1(\beta x)$ or $f(x) = \frac{1}{\alpha} g_1(\beta x)$ or $f(x) = \frac{1}{\alpha} f^*_\delta(\beta x)$ for some $\alpha, \beta, \delta > 0$.*

Remark 1.3

1. The four classes of LWMY functions are

$$\mathcal{F}_1 = \{\alpha/x;\, \alpha > 0\}, \quad \mathcal{F}_2 = \{\frac{1}{\alpha} f_1(\beta x);\, \alpha, \beta > 0\},$$

$$\mathcal{F}_3 = \{\frac{1}{\alpha} g_1(\beta x);\, \alpha, \beta > 0\}, \quad \mathcal{F}_4 = \{\frac{1}{\alpha} f^*_\delta(\beta x);\, \alpha, \beta > 0\}.$$

2. It is clear that if f is a LWMY function, then the functions f^{-1} and $x \mapsto \frac{1}{\alpha} f(\beta x)$, $\alpha, \beta > 0$ are LWMY functions. Consequently, $x \mapsto \frac{1}{\alpha} f_1(\beta x)$ is a LWMY function if and only if $x \mapsto \frac{1}{\alpha} g_1(\beta x)$ is a LWMY function.

3. Note that $x \mapsto \alpha/x$ and f^*_δ are self-reciprocal.

The first case corresponds to the one studied by Matsumoto-Yor (2001) and Letac-Wesolowski (2000). The two last authors proved that if $x \mapsto 1/x$ is a LWMY function with respect to (X, Y), then the laws of X and Y are GIG and gamma distributions respectively (we partially recover this case in Appendix 5.1). Therefore, in the sequel we will focus on the three new cases : either $f = f_1$ or $f = g_1$ or $f = f^*_\delta$ and in each case we determine the laws of the related random variables.

1.1 The cases $f = g_1$ and $f = f_1$

a) Consider the gamma distribution

$$\gamma(\lambda, c)(dx) = \frac{c^\lambda}{\Gamma(\lambda)} x^{\lambda-1} e^{-cx} \mathbf{1}_{(0,\infty)}(x) dx, \quad (\lambda, c > 0)$$

and the beta distribution

$$\mathrm{Beta}(a, b)(dx) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1} \mathbf{1}_{\{0<x<1\}} dx, \quad (a, b > 0).$$

Consider (see for instance Ng and Kotz, 1995, or Nagar and Gupta, 2002 and references therein) the *Kummer distribution of type 2* :

$$K^{(2)}(a, b, c) := \alpha(a, b, c) x^{a-1} (1+x)^{-a-b} e^{-cx} \mathbf{1}_{(0,\infty)}(x) dx, \ a, c > 0, \ b \in \mathbb{R} \tag{1.18}$$

where $\alpha(a, b, c)$ is a normalizing constant.

Associated with a couple (X, Y) of positive r.v.'s consider

$$(U, V) := T_{f_1}(X, Y) = \left(\frac{1}{e^{X+Y} - 1}, \frac{1}{e^X - 1} - \frac{1}{e^{X+Y} - 1} \right). \tag{1.19}$$

In Theorems 1.4 and 1.6 below we suppose that all r.v.'s have positive and twice differentiable densities.

First we consider the case $f = f_1$. We determine the distributions of X and Y such that f_1 is a LWMY function associated to (X, Y).

Theorem 1.4 1. *Consider two positive and independent random variables X and Y. The random variables U and V defined by (1.19) are independent if and only if the densities of Y and X are respectively*

$$p_Y(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}(1-e^{-y})^{b-1}e^{-ay}\mathbf{1}_{\{y>0\}} \tag{1.20}$$

$$\begin{aligned} p_X(x) &= \alpha(a+b, c, -a)e^{-(a+b)x}(1-e^{-x})^{-b-1} \\ &\quad \exp\left(-c\frac{e^{-x}}{1-e^{-x}}\right)\mathbf{1}_{\{x>0\}}. \end{aligned} \tag{1.21}$$

where a, b and c are constants such that $a, b, c > 0$ and $\alpha(a+b, c, -a)$ is the constant of Equation (1.18). Thus the law of Y is the image of the Beta(a, b) distribution by the transformation $z \in (0,1) \mapsto -\log z \in (0, \infty)$, while the law of the variable $f_1(X)$ is $K^{(2)}(a+b, -b, c)$ (cf Equation (1.18)).

2. *If 1. holds then $U \sim K^{(2)}(a, b, c)$ and $V \sim \gamma(b, c)$.*

The proof of Theorem 1.4 will be given in Section 4.

Remark 1.5 Since $g_1 = f_1^{-1}$, item 2. of Remark 1.3 and Theorem 1.4 imply that the r.v.'s associated with the LWMY function g_1 are the r.v.'s U and V distributed as in item 2. of Theorem 1.4.

b) As suggest identities (1.20) and (1.21) it is actually possible to simplify Theorem 1.4.

Since $T_{g_1} = T_{f_1}^{-1}$, then

$$(X, Y) = T_{g_1}(U, V) = \left(\log\left(\frac{1+U+V}{U+V}\right), \log\left(\frac{1+U}{U}\right) - \log\left(\frac{1+U+V}{U+V}\right)\right). \tag{1.22}$$

As shows (1.22) it is useful to introduce

$$(U', V') = \left(\frac{1+\frac{1}{U+V}}{1+\frac{1}{U}}, U+V\right). \tag{1.23}$$

Obviously the correspondence $(U, V) \mapsto (U', V')$ is one-to one:

$$(U, V) = \left(\frac{U'V'}{V'+1-U'V'}, \frac{V'(V'+1)(1-U')}{V'+1-U'V'}\right). \tag{1.24}$$

Furthermore, (X, Y) can be easily expressed in terms of (U', V'):

$$X = \log(1 + 1/V') \text{ and } Y = -\log U'. \tag{1.25}$$

Since it is easy to determine the density function of $\phi(\xi)$ where ϕ is differentiable and bijective, then Theorem 1.4 and its analogue related to $f = g_1$ (cf Remark 1.5) are equivalent to Theorem 1.6 below.

Theorem 1.6 a) *Let U' and V' be two positive and independent random variables. The r.v.'s U and V defined by (1.24) are independent if only if there exist some constants a, b, c such that*

$$U' \sim \mathrm{Beta}(a,b) \quad \text{and} \quad V' \sim K^{(2)}(a+b,-b,c). \tag{1.26}$$

b) *Let U and V be two positive and independent random variables. The r.v.'s U' and V' defined by (1.23) are independent if only if there exist some constants a, b, c such that*

$$U \sim K^{(2)}(a,b,c) \quad \text{and} \quad V \sim \gamma(b,c). \tag{1.27}$$

Under (1.27), $U' \sim \mathrm{Beta}(a,b)$ and $V' \sim K^{(2)}(a+b,-b,c)$.

c) *If one of these equivalent conditions holds, then $U \sim K^{(2)}(a,b,p)$ and $V \sim \gamma(b,c)$.*

Let us formulate an easy consequence of Theorem 1.6.

Theorem 1.7 *For any $a,b,c>0$, the transformation $(u,v) \mapsto (\frac{1+\frac{1}{u+v}}{1+\frac{1}{u}},\ u+v)$ maps the probability measure $K^{(2)}(a,b,c) \otimes \gamma(b,c)$ to the probability measure $\mathrm{Beta}(a,b) \otimes K^{(2)}(a+b,-b,c)$. In particular:*

$$K^{(2)}(a,b,c) * \gamma(b,c) = K^{(2)}(a+b,-b,c). \tag{1.28}$$

Remark 1.8

1. Note that (1.28) may be regarded as an analogue of (0.2).
2. Theorem 1.6 and the results of Letac and Wesolowski (2000) suggest to ask whether it is possible to have an "algebraic" proof of the following part of either item a) or item b) of Theorem 1.6: if U and V, U' and V' are independent, then (1.26) and (1.27) hold.

1.2 The case $f = f_\delta^*$

Recall that f_δ^* has been defined by (1.17). Due to the form of f_δ^*, a change of variables allows to simplify the search of independent r.v.'s X and Y such that the two components of $T_{f_\delta^*}(X,Y)$ are independent.
For any decreasing and bijective function $f:(0,\infty)\to(0,\infty)$ we define

$$\overline{f}(x) = \exp\{-f(-\log x)\}, \quad x \in (0,1), \tag{1.29}$$

$$T_f^m(x,y) = \left(f(xy), \frac{f(x)}{f(xy)}\right), \quad x,\ y \in (0,1). \tag{1.30}$$

Observe that $\overline{f}$ is one-to-one from $(0,1)$ onto $(0,1)$, T_f^m is one-to-one from $(0,1)^2$ onto $(0,1)^2$ and

$$\left(T_f^m\right)^{-1} = T_{f^{-1}}^m. \tag{1.31}$$

Definition 1.9 *Let X and Y be two independent and $(0,1)$-valued random variables. We say that a decreasing and bijective function $f:(0,1)\to(0,1)$ is a multiplicative LWMY function with respect to (X,Y) if the r.v.'s $U^m := f(XY)$ and $V^m := \frac{f(X)}{f(XY)}$ are independent.*

Remark 1.10 *For any random vector (X,Y) in $(0,\infty)^2$ we consider*

$$X' = e^{-X},\ Y' = e^{-Y}.$$

Then f is a LWMY function with respect to (X,Y) if and only if $\overline{f}$ is a multiplicative LWMY function with respect to (X',Y').

The change of variable $x' = e^{-x}$ is very convenient since the function

$$\phi_\delta(x) := \overline{f_\delta^*}(x) = \frac{1-x}{1+(\delta-1)x},\quad x\in(0,1) \tag{1.32}$$

is homographic.
Note that $\overline{f_\delta^*}:(0,1)\to(0,1)$ is bijective, decreasing and self-reciprocal.
First, let us determine the distribution of the couple (X',Y') of r.v.'s such that ϕ_δ is a multiplicative LWMY function with respect to (X',Y').

For $a,b,\alpha>0$ and $c\in\mathbb{R}$ consider the probability measure

$$\beta_\alpha(a,b;c)(dx) = k_\alpha(a,b;c)x^{a-1}(1-x)^{b-1}(\alpha x+1-x)^c\mathbf{1}_{(0,1)}(x)dx. \tag{1.33}$$

Note that if $c=0$, then $\beta_\alpha(a,b;c)=\mathrm{Beta}(a,b)$.

Theorem 1.11 *Let X' and Y' be two independent random variables valued in $(0,1)$. Consider*

$$(U^m,V^m) = T^m_{\phi_\delta}(X',Y') = \left(\frac{1-X'Y'}{1+(\delta-1)X'Y'},\ \frac{1-X'}{1+(\delta-1)X'}\frac{1+(\delta-1)X'Y'}{1-X'Y'}\right)$$

for fixed $\delta>0$.
Then, U^m and V^m are independent if and only if there exist $a,b,\lambda>0$ such that

$$X'\sim\beta_\delta(a+b,\lambda;-\lambda-b),\ Y'\sim\mathrm{Beta}(a,b). \tag{1.34}$$

If this condition holds, then

$$U^m\sim\beta_\delta(\lambda+b,a;-a-b),\ V^m\sim\mathrm{Beta}(\lambda,b). \tag{1.35}$$

In the case $\delta=1$, Theorem 1.11 takes a very simple form.

Proposition 1.12 *Let X' and Y' be two independent random variables valued in $(0,1)$. Then*

$$U^m = 1-X'Y',\ V^m = \frac{1-X'}{1-X'Y'}$$

are independent if and only if there exist $a,b,\lambda>0$ such that

$$X'\sim\mathrm{Beta}(a+b,\lambda)\ \text{and}\ Y'\sim\mathrm{Beta}(a,b).$$

If one of these conditions holds, then

$$U^m\sim\mathrm{Beta}(\lambda+b,a),\ \text{and}\ V^m\sim\mathrm{Beta}(\lambda,b).$$

Remark 1.13 When $X' \sim \text{Beta}(a+b,\lambda)$ and $Y' \sim \text{Beta}(a,b)$ it can be proved that U^m and V^m are independent using the well-known property: if Z and Z' are independent, $Z \sim \gamma(a,1)$ and $Z' \sim \gamma(b,1)$ then $R := \frac{Z}{Z+Z'}$ and $Z+Z'$ are independent and $R \sim \text{Beta}(a,b)$ and $Z+Z' \sim \gamma(a+b,1)$ (see for instance Yor, 1989).

According to Remark 1.10, f_δ^* is a LWMY function with respect to (X,Y) if and only if ϕ_δ is a multiplicative LWMY function with respect to $(X',Y') = (e^{-X}, e^{-Y})$. Therefore, a classical change of variables allows to deduce that Theorem 1.11 is equivalent to Theorem 1.14 below:

Theorem 1.14 1. *Consider two positive and independent random variables X and Y. The random variables $U = f_\delta^*(X+Y)$, $V = f_\delta^*(X) - f_\delta^*(X+Y)$ are independent if and only if the densities of Y and X are respectively*

$$p_Y(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}(1-e^{-y})^{b-1}e^{-ay}\mathbf{1}_{\{y>0\}} \tag{1.36}$$

$$\begin{aligned} p_X(x) &= k_\delta(a+b,\lambda,-\lambda-b)e^{-(a+b)x}(\delta e^{-x}+1-e^{-x})^{-\lambda-b} \\ &\quad \times(1-e^{-x})^{\lambda-1}\mathbf{1}_{x>0} \end{aligned} \tag{1.37}$$

where $a,b>0$, $\lambda \in \mathbb{R}$ and $k_\delta(a+b,\lambda,-\lambda-b)$ is the normalizing factor (cf (1.33)). Thus e^{-Y} is Beta(a,b) distributed and e^{-X} is $\beta_\delta(a+b,\lambda,-\lambda-b)$ distributed.

2. *If 1. holds then the densities of U and V are respectively*

$$\begin{aligned} p_U(u) &= k_\delta(\lambda+b,a;-a-b)e^{-u(\lambda+b)}(1-e^{-u})^{a-1} \\ &\quad \times(1+(\delta-1)e^{-u})^{-a-b}\mathbf{1}_{u>0}, \end{aligned} \tag{1.38}$$

$$p_V(v) = e^{-\lambda v}(1-e^{-v})^{b-1}\,\mathbf{1}_{v>0}. \tag{1.39}$$

The proof of Theorem 1.14 is similar to that of Theorem 1.4 and has been postponed to Appendix 5.3.

2 A necessary and sufficient condition for LWMY functions and related densities

Theorem 2.1 *Let X and Y be two independent and positive random variables whose densities p_X and p_Y are positive and twice differentiable. Define $\phi_X = \log p_X$ and $\phi_Y = \log p_Y$. Consider a decreasing function $f : (0,\infty) \mapsto (0,\infty)$, three times differentiable. Then f is a LWMY function with respect to (X,Y) if and only if*

$$\begin{aligned} &\phi_X''(x) - \phi_X'(x)\frac{f''(x)}{f'(x)} + \phi_Y''(y)f'(x)\left(\frac{1}{f'(x)} - \frac{1}{f'(x+y)}\right) \\ &+\phi_Y'(y)\frac{f''(x)}{f'(x)} + \frac{2(f''(x))^2 - f'''(x)f'(x)}{f'(x)^2} = 0, \quad x,y>0. \end{aligned} \tag{2.1}$$

Proof :

1) Consider the transformation $T_f : (0,\infty)^2 \to (0,\infty)^2$; $(x,y) \mapsto (f(x+y), f(x) - f(x+y))$. Let $g = f^{-1}$ and $(U,V) = T_f(X,Y)$. By formula (1.13), $(X,Y) = T_g(U,V)$. X and Y being independent, the density of (U,V) is

$$p_{(U,V)}(u,v) = p_X(g(u+v))\, p_Y(g(u) - g(u+v))\, |J(u,v)| 1_{u,v>0} \tag{2.2}$$

where J is the Jacobian of the transformation T_f. Since $x = g(u+v)$ and $y = g(u) - g(u+v)$ we have

$$J(u,v) = \det \begin{pmatrix} g'(u+v) & g'(u+v) \\ g'(u) - g'(u+v) & -g'(u+v) \end{pmatrix} = -g'(u+v)g'(u).$$

g being monotone we have $|J(u,v)| = g'(u+v)g'(u)$, so that

$$p_{(U,V)}(u,v) = p_X(g(u+v))\, p_Y(g(u) - g(u+v))\, g'(u+v)g'(u). \tag{2.3}$$

2) Let $H : (0,\infty)^2 \to \mathbb{R}$ be a function having second partial derivatives. It is easy to prove that the function H decomposes as

$$H(u,v) = h_1(u) + h_2(v), \quad u,v > 0,$$

if and only if $\frac{\partial^2 H}{\partial u \partial v} = 0$.

3) According to the former item, the variables U and V are independent if and only if the function $H = \log p_{(U,V)}$ satisfies $\frac{\partial^2 H}{\partial u \partial v} = 0$. By Equation (2.3) we have

$$\frac{\partial H}{\partial v} = \phi_X'(g(u+v))\, g'(u+v) - \phi_Y'(g(u) - g(u+v))\, g'(u+v) + \frac{g''}{g'}(u+v),$$

$$\begin{aligned} \frac{\partial^2 H}{\partial u \partial v} &= \phi_X''(g(u+v))\, (g'(u+v))^2 + \phi_X'(g(u+v))\, g''(u+v) \\ &\quad - \phi_Y''(g(u) - g(u+v))\, g'(u+v)\, [g'(u) - g'(u+v)] \\ &\quad - \phi_Y'(g(u) - g(u+v))\, g''(u+v) + \frac{g'''g' - (g'')^2}{(g')^2}(u+v). \end{aligned}$$

Writing $x = g(u+v)$ and $y = g(u) - g(u+v)$, we have $u = f(x+y)$, $v = f(x) - f(x+y)$ and

$$\begin{aligned} \frac{\partial^2 H}{\partial u \partial v} &= \phi_X''(x)[g'(f(x))]^2 + \phi_X'(x) g''(f(x)) \\ &\quad - \phi_Y''(y) g'(f(x))\, [g'(f(x+y)) - g'(f(x))] \\ &\quad - \phi_Y'(y) g''(f(x)) + \frac{g'''g' - (g'')^2}{(g')^2}(f(x)) = 0. \end{aligned} \tag{2.4}$$

Differentiating twice the relation $g(f(x)) = x$, we obtain

$$g'(f(x)) = \frac{1}{f'(x)}, \quad (x > 0) \tag{2.5}$$

$$g''(f(x)) = -\frac{f''(x)}{f'(x)^3} \quad (x > 0). \tag{2.6}$$

A differentiation of the latter equality yields

$$g'''(f(x)) = -\frac{f'''(x)f'(x) - 3f''(x)^2}{f'(x)^5}.$$

Using this equality and (2.5) and (2.6) we have

$$\begin{aligned} \frac{g'''g' - (g'')^2}{(g')^2}(f(x)) &= \left[-\left(\frac{f'''(x)f'(x) - 3f''(x)^2}{f'(x)^5} \right) \frac{1}{f'(x)} - \frac{f''(x)^2}{f'(x)^6} \right] f'(x)^2 \\ &= \frac{2f''(x)^2 - f'''(x)f'(x)}{f'(x)^4}. \end{aligned} \tag{2.7}$$

Plugging Equations (2.5), (2.6) and (2.7) into Equation (2.4) one gets

$$\begin{aligned} \frac{\partial^2 H}{\partial u \partial v} &= \frac{\phi_X''(x)}{f'(x)^2} - \frac{\phi_X'(x)f''(x)}{f'(x)^3} - \frac{\phi_Y''(y)}{f'(x)} \left(\frac{1}{f'(x+y)} - \frac{1}{f'(x)} \right) \\ &\quad + \frac{\phi_Y'(y)f''(x)}{f'(x)^3} + \frac{2f''(x)^2 - f'''(x)f'(x)}{f'(x)^4} \end{aligned}$$

and (2.1) follows by multiplying the above identity by $f'(x)^2$.

□

3 The set of all possible "smooth" LWMY functions

We restrict ourselves to *smooth* LWMY functions f, i.e. satisfying

$$f : (0, \infty) \to (0, \infty) \text{ is bijective and decreasing,} \tag{3.1}$$

$$f \text{ is three times differentiable,} \tag{3.2}$$

$$F(x) = \sum_{n \geq 1} a_n x^n, \ \forall x > 0. \tag{3.3}$$

where $F := 1/f'$.
According to (3.1), $f'(0_+) = -\infty$. This implies $F(0_+) = 0$ and explains why the series in (3.3) starts with $n = 1$.

The goal of this section is to prove half of Theorem 1.2: if f is a smooth LWMY function, then f belongs to one of the four classes $\mathcal{F}_1, \ldots, \mathcal{F}_4$ introduced in Remark 1.3. First, we characterize in Theorem 3.1 all possible functions F. Second, we determine the associated functions f (see Theorem 3.8).

Theorem 3.1 *Suppose that f is a smooth LWMY function and that the assumptions of Theorem 2.1 are satisfied.*

1. *If $F'(0_+) = 0$, then $a_2 < 0$ and*

$$F(x) = \begin{cases} \frac{a_2^2}{6a_4}\left(\cosh\left(x\sqrt{\frac{12a_4}{a_2}}\right) - 1\right) & \text{if } a_4 < 0 \\ a_2 x^2 & \text{otherwise.} \end{cases} \tag{3.4}$$

2. *If $F'(0_+) \neq 0$, then*

$$F(x) = \begin{cases} \frac{a_1 a_2}{3a_3}\left[\cosh\left(x\sqrt{\frac{6a_3}{a_1}}\right) - 1\right] + a_1\sqrt{\frac{a_1}{6a_3}}\sinh\left(x\sqrt{\frac{6a_3}{a_1}}\right) & \text{if } a_1 a_3 > 0 \\ a_1 x + a_2 x^2 & \text{otherwise.} \end{cases} \tag{3.5}$$

Remark 3.2 *Unsurprisingly, the case $F(x) = a_2 x^2$ corresponds to $f(x) = -\frac{1}{a_2}\frac{1}{x}$, i.e. the case considered by Matsumoto-Yor and Letac-Wesolowski.*

Throughout this subsection we suppose that f satisfies (3.1), (3.2), (3.3) and that the assumptions of Theorem 2.1 are fulfilled. For simplicity of statement of results below we do not repeat these conditions.

Recall that ϕ_Y is the logarithm of the density of Y. Let us introduce

$$h := \phi_Y' \tag{3.6}$$

Lemma 3.3 1. *There exists a function $\lambda : (0, \infty) \to \mathbb{R}$ such that*

$$F(x+y) = \frac{\lambda(x) - h(y)F'(x)}{h'(y)} + F(x). \tag{3.7}$$

2. *F satisfies*

$$F(y) = \frac{\lambda(0_+) - h(y)F'(0_+)}{h'(y)}. \tag{3.8}$$

Remark 3.4 *Suppose that we have been able to determine F. Then, $h = \phi_Y'$ solves the linear ordinary differential equation (3.8) and can therefore be determined. The remaining function ϕ_X is obtained by solving Equation (2.1).*

Proof of Lemma 3.3 :
Using (3.6) and $F = 1/f'$ in Equation (2.1), we obtain

$$c(x) = h(y)\frac{F'(x)}{F(x)} + h'(y)\frac{1}{F(x)}(F(x+y) - F(x))$$

where $c(x)$ depends only on x. Multiplying both sides by $F(x)$ and taking the y-derivative leads to

$$0 = F'(x)h'(y) + [F(x+y) - F(x)]h''(y) + h'(y)F'(x+y).$$

Fix $x > 0$. Then $\theta(y) := F(x+y)$ is a solution of the differential equation in y:

$$0 = F'(x)h'(y) + (\theta(y) - F(x))h''(y) + h'(y)\theta'(y). \tag{3.9}$$

A solution of the related homogeneous equation in y is $\frac{\rho}{h'(y)}$ where ρ is a constant. It is easy to prove that $y \mapsto -F'(x)h(y) + F(x)h'(y)$ solves (3.9). Thus, the general solution of (3.9) is

$$\theta(y) = \frac{1}{h'(y)} \left[\lambda(x) - F'(x)h(y) + F(x)h'(y)\right].$$

Since $\theta(y) = F(x+y)$, (3.7) follows.

According to (3.3), $F(0_+)$ and $F'(0_+)$ exist. Therefore, taking the limit $x \to 0_+$ in (3.7) implies both the existence of $\lambda(0_+)$ and relation (3.8).

□

The following lemma shows that the function F (and thus f) solves a self-contained equation in which h, and thereby the densities of X and Y, are not involved.

Lemma 3.5 *F solves the delay equations :*

$$F(x+y) = \frac{F(y)[\lambda(x) - h(y)F'(x)]}{\lambda(0_+) - h(y)F'(0_+)} + F(x) \quad (x, y > 0) \tag{3.10}$$

$$F'(x+y) = \frac{F'(y) + F'(0_+)}{F(y)}[F(x+y) - F(x)] - F'(x) \quad (x, y > 0). \tag{3.11}$$

Proof:
By (3.8) we have

$$h'(y) = \frac{\lambda(0_+) - h(y)F'(0_+)}{F(y)}.$$

Equation (3.10) then follows by rewriting Equation (3.7) and replacing $h'(y)$ with the expression above.
We differentiate (3.10) in y and use the fact that $\lambda(0_+) - h(y)F'(0_+) = h'(y)F(y)$ to obtain :

$$F'(x+y) = [F'(y) + F'(0_+)]\frac{\lambda(x) - h(y)F'(x)}{F(y)h'(y)} - F'(x).$$

By (3.7) we have

$$\frac{\lambda(x) - h(y)F'(x)}{F(y)h'(y)} = \frac{F(x+y) - F(x)}{F(y)}$$

and this gives (3.11).

□

Remark 3.6 We can see (3.11) as a scalar neutral delay differential equation. Indeed, set $t = x + y$ and consider $y > 0$ as a fixed parameter. Then (3.11) becomes:

$$F'(t) = a(F(t) - F(t-y)) - F'(t-y), \quad t \geq y, \tag{3.12}$$

where $a := \frac{F'(y)+F'(0_+)}{F(y)}$.
Replacing $F(t)$ in (3.12) with $e^{at}G(t)$ leads to:

$$G'(t) + e^{-ay}G'(t-y) + 2ae^{-ay}G(t-y) = 0, \quad t \geq y. \tag{3.13}$$

Equation (3.13) is called a neutral delay differential equation (cf for instance, Section 6.1, in Györi and Ladas, 1991). These equations have been intensively studied but the authors only focused on the asymptotic behaviour of the solution as $t \to \infty$. Unfortunately, these results give no help to solve explicitly either (3.11) or (3.13).

Lemma 3.7 *For all integers $k \geq 0$ and $l \geq 1$, we have*

$$\sum_{m=0}^{l-1}(l-2m+1)C^k_{l-m+1+k}a_{l-m+1+k}a_m = (l-2)(k+1)a_{k+1}a_l + a_1 a_{l+k}C^k_{l+k}, \tag{3.14}$$

$$C^k_{k+3}a_{k+3}a_1 = (k+1)a_{k+1}a_3, \tag{3.15}$$

$$2C^k_{k+4}a_{k+4}a_1 + C^k_{k+3}a_{k+3}a_2 - C^k_{k+2}a_{k+2}a_3 - 2(k+1)a_{k+1}a_4 = 0, \tag{3.16}$$

where $C^p_n = \frac{n!}{(n-p)!p!}$.

Proof:
Obviously Equation (3.11) is equivalent to:

$$F'(x+y)F(y) = F'(y)F(x+y) - F'(y)F(x) - F(y)F'(x) + F'(0_+)F(x+y) - F'(0_+)F(x). \tag{3.17}$$

Using the asymptotic expansion (3.3) of F we can develop each term in (3.17) as a series with respect to x and y. Then, identifying the series on the right-hand side and the left-hand side we get (3.14)-(3.16). Since this approach only needs usual skill in analysis the details are provided in Appendix 6.2.

□

Proof of part 1. of Theorem 3.1

We suppose that $F'(0_+) = 0$.
Since $a_1 = F'(0_+) = 0$, we necessarily have $a_2 \neq 0$. Indeed, if $a_2 = 0$ then, by (3.16) with $k = 1$, we would have $-3a_3^2 - 4a_2a_4 = 0$, i.e. $a_3 = 0$, and using again (3.16) with $k = 3$ would imply $a_4 = 0$ and finally $a_k = 0$ for every $k \geq 0$, which is a contradiction because, by definition, $F = 1/f'$ does not vanish.

So, we have $a_1 = 0$ and $a_2 \neq 0$. Equation (3.15) with $k = 1$ reads $4a_4a_1 = 2a_2a_3$, which implies $a_3 = 0$. Applying (3.15) to $k = 2n$ provides, by induction on n, $a_{2n+1} = 0$ for every $n \geq 0$.

Therefore, Equation (3.16) reduces to

$$(k+3)(k+2)(k+1)a_{k+3}a_2 = 12(k+1)a_{k+1}a_4, \quad (k \geq 0)$$

i.e.

$$a_{k+3} = \frac{12a_4}{a_2}\frac{1}{(k+3)(k+2)}a_{k+1}.$$

This leads to

$$a_{2k} = \left(\frac{12a_4}{a_2}\right)^{k-1}\frac{2}{(2k)!}a_2, \qquad k \geq 1. \tag{3.18}$$

Then, $F(x) = a_2x^2$ if $a_4 = 0$ and if $a_4 \neq 0$ we have

$$F(x) = \sum_{k\geq 1}\left(\frac{12a_4}{a_2}\right)^{k-1}\frac{2}{(2k)!}a_2x^{2k}.$$

If $a_4a_2 < 0$, then

$$F(x) = \frac{a_2^2}{6a_4}\left[\cos\left(x\sqrt{\frac{-12a_4}{a_2}}\right) - 1\right].$$

This implies $F(2\pi\sqrt{\frac{-12a_4}{a_2}}) = 0$ which is impossible since $F(x) = 1/f'(x) < 0$. Consequently,

$$F(x) = \frac{a_2^2}{6a_4}\left[\cosh\left(x\sqrt{\frac{12a_4}{a_2}}\right) - 1\right].$$

□

Proof of part 2. of Theorem 3.1

Using Equation (3.15) we have, for all $k \geq 0$, $(k+3)(k+2)a_{k+3}a_1 = 6a_{k+1}a_3$. Since $a_1 \neq 0$, Equation (3.15) is equivalent to

$$a_{k+3} = \frac{6a_3}{a_1}\frac{1}{(k+3)(k+2)}a_{k+1}.$$

As a result,

$$a_{2k+1} = \left(\frac{6a_3}{a_1}\right)^{k}\frac{1}{(2k+1)!}a_1, \qquad (k \geq 0),$$

$$a_{2k} = \left(\frac{6a_3}{a_1}\right)^{k-1}\frac{2}{(2k)!}a_2, \qquad (k \geq 1).$$

As a consequence,

$$F(x) = \sum_{k\geq 1}\left(\frac{6a_3}{a_1}\right)^{k-1}\frac{2}{(2k)!}a_2x^{2k} + \sum_{k\geq 0}\left(\frac{6a_3}{a_1}\right)^{k}\frac{1}{(2k+1)!}a_1x^{2k+1}.$$

If $a_3 = 0$ then it follows from Equation (3.15) that $a_k = 0$ for every $k \geq 3$. If $a_3 \neq 0$ and $a_1a_3 > 0$ we have

$$\sum_{k\geq 1}\left(\frac{6a_3}{a_1}\right)^{k-1}\frac{2}{(2k)!}a_2x^{2k} = \frac{a_2a_1}{3a_3}\left[\cosh\left(x\sqrt{\frac{6a_3}{a_1}}\right) - 1\right]$$

and

$$\sum_{k\geq 0}\left(\frac{6a_3}{a_1}\right)^k \frac{1}{(2k+1)!}a_1x^{2k+1} = a_1\sqrt{\frac{a_1}{6a_3}}\sinh\left(x\sqrt{\frac{6a_3}{a_1}}\right).$$

If $a_1a_3 < 0$ then

$$F(x) = a_1\left(\frac{a_2}{3a_3}\left[\cos\left(x\sqrt{\frac{-6a_3}{a_1}}\right)-1\right]+\sqrt{\frac{-a_1}{6a_3}}\sin\left(x\sqrt{\frac{-6a_3}{a_1}}\right)\right).$$

In particular, $F(2\pi\sqrt{\frac{-a_1}{6a_3}}) = 0$. This is impossible since $F(x) = 1/f'(x) < 0$.

□

Now, in each case of Theorem 3.1 we compute the corresponding f associated with F via the relation $F = 1/f'$. Recall that we restrict ourselves to functions f satisfying (3.1)-(3.3) and work under the assumptions of Theorem 2.1.

Theorem 3.8 1. *If* $F(x) = a_2x^2$ *then* $f(x) = \frac{1}{a_2x}$.

2. *If* $F(x) = \alpha(\cosh\beta x - 1), \alpha, \beta > 0$, *then* $f(x) = \frac{2}{\alpha\beta}f_1(\beta x)$.

3. *If* $F(x) = a_1x + a_2x^2$ *then* $f(x) = -\frac{1}{a_1}g_1(\frac{a_2}{a_1}x)$.

4. *If* $F(x) = \dfrac{a_1a_2}{3a_3}\left[\cosh\left(x\sqrt{\dfrac{6a_3}{a_1}}\right)-1\right]+a_1\sqrt{\dfrac{a_1}{6a_3}}\sinh\left(x\sqrt{\dfrac{6a_3}{a_1}}\right)$ *then*

$$f(x) = -\frac{1}{\beta\gamma}\log\left(\frac{e^{\beta x}+\delta-1}{e^{\beta x}-1}\right),$$

where $\alpha = \frac{a_1a_2}{3a_3}$, $\beta = \sqrt{\frac{6a_3}{a_1}}$ *and* $\gamma = a_1\sqrt{\frac{a_1}{6a_3}}$.

Proof:
1) The first case is obvious.
2) We have

$$f'(x) = \frac{1}{F(x)} = -\frac{1}{\alpha(\cosh\beta x - 1)}, \quad x > 0, \beta > 0, \alpha > 0.$$

Since $f(0_+) = +\infty$ and $f(+\infty) = 0^+$, integrating the previous identity gives

$$f(x) = \frac{2}{\alpha\beta}\frac{1}{e^{\beta x}-1} = \frac{2}{\alpha\beta}f_1(\beta x),$$

where $f_1(x) = \frac{1}{e^x-1}$.
3) Recall (cf (1.16)) that $g_1 = f_1^{-1}$. Since $g_1'(x) = -\frac{1}{x(x+1)}$, it can be easily proved that

the associated LWMY function is $f(x) = -\frac{1}{a_1} g_1(\frac{a_2}{a_1} x)$.
4) Due to our choice of α, β and γ we have

$$\begin{aligned} f'(x) &= \frac{1}{F(x)} \\ &= \frac{1}{\alpha(\cosh \beta x - 1) + \gamma \sinh \beta x} \\ &= \frac{e^{\beta x}}{\gamma(e^{\beta x} - 1)} - \frac{(\alpha + \gamma) e^{\beta x}}{\gamma[(\alpha + \gamma)e^{\beta x} - \alpha + \gamma]}. \end{aligned} \tag{3.19}$$

As a consequence, by integration,

$$f(x) = \frac{1}{\beta\gamma} \log(e^{\beta x} - 1) - \frac{1}{\beta\gamma} \log |(\alpha + \gamma)e^{\beta x} - \alpha + \gamma| + C$$

where C is a constant.
Using (3.19) we have : $f'(x) \sim \frac{1}{\alpha\beta} \frac{1}{x}$ as $x \to 0^+$, $f'(x) \sim \frac{2}{\alpha+\gamma} e^{-\beta x}$ as $x \to +\infty$. This implies that $\gamma < 0$ and $\alpha + \gamma < 0$. Setting $\delta = \dfrac{2\gamma}{\alpha + \gamma}$ we have $\delta > 0$, $\dfrac{\gamma - \alpha}{\alpha + \gamma} = \delta - 1$ and

$$f(x) = -\frac{1}{\beta\gamma} \log \left(\frac{e^{\beta x} + \delta - 1}{e^{\beta x} - 1} \right) + C'.$$

Since $f(\infty) = 0$ we have $C' = 0$. As a result, $f(x) = -\frac{1}{\beta\gamma} f_\delta^*(\beta x)$.

□

4 Proof of Theorem 1.4

Recall that f_1, g_1 and f_δ^* have been defined by equations (1.15), (1.16) and (1.17) respectively. In Theorem 3.8 we have proved that if f is a LWMY function, then $f \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$, where these four classes have been introduced in Remark 1.3. The class $\mathcal{F}_1$ corresponds to the Matsumoto-Yor property. Indeed, we prove in Subsection 6.1 that $x \mapsto 1/x$ is a LWMY function associated to GIG and gamma distributions. The result has been already obtained by Letac and Wesolowski (2001) under weaker assumptions than ours.

Recall that $\phi_Y = \log p_Y$, $h = \phi'_Y$ and $F'(0_+) = 0$. It is easy to deduce from (3.8) that there exist constants λ and c_1 such that

$$\begin{aligned} h(y) &= \lambda f(y) + c_1 \\ &= \frac{\lambda}{e^y - 1} + c_1 \\ &= \frac{\lambda e^y}{e^y - 1} + c_1 - \lambda. \end{aligned}$$

This implies the existence of a constant d such that

$$\phi_Y(y) = \lambda \log(e^y - 1) + (c_1 - \lambda)y + d.$$

Setting $M = e^d$, we have by integration, for all $y > 0$,

$$\begin{aligned} p_Y(y) &= M(e^y - 1)^\lambda e^{(c_1-\lambda)y} \\ &= M(1 - e^{-y})^\lambda e^{c_1 y}. \end{aligned} \tag{4.1}$$

To give more information on the normalizing constant M, one observes, for $a = -c_1$ and $b = \lambda + 1$, that

$$\int_0^\infty M(1 - e^{-y})^{b-1} e^{-ay}\, dy = M \int_0^1 (1-u)^{b-1} u^{a-1}\, du$$

which implies that $a > 0$, $b > 0$ and

$$M = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}.$$

This proves (1.20).

To find the density of X we come back to Equation (2.1) and compute each of its terms.

We have $f'(x) = \frac{-e^x}{(e^x-1)^2}$, $f''(x) = \frac{e^{2x}+e^x}{(e^x-1)^3}$, $f'''(x) = -\frac{e^{3x}+4e^{2x}+e^x}{(e^x-1)^4}$, so that $\frac{f'(x)}{f'(x+y)} = \frac{e^{-y}(e^{x+y}-1)^2}{(e^x-1)^2}$ and $\frac{f''(x)}{f'(x)} = -\frac{e^x+1}{e^x-1}$. Calculations yield

$$\frac{2(f''(x))^2 - f'''(x)f'(x)}{f'(x)^2} = \frac{e^{2x}+1}{(e^x-1)^2}. \tag{4.2}$$

Moreover,

$$\begin{aligned} -\phi_Y'(y)\frac{f''(x)}{f'(x)} + \phi_Y''(y)\left(\frac{f'(x)}{f'(x+y)} - 1\right) &= \left(\frac{\lambda}{e^y-1} + c_1\right)\frac{e^x+1}{e^x-1} \\ &\quad - \frac{\lambda e^y}{(e^y-1)^2}\left(\frac{e^{-y}(e^{x+y}-1)^2}{(e^x-1)^2} - 1\right) \\ &= \frac{(c_1-\lambda)e^{2x} - c_1}{(e^x-1)^2}. \end{aligned} \tag{4.3}$$

Equation (2.1) can then be writen, using (4.2) and (4.3):

$$\begin{aligned} \phi_X''(x) + \frac{e^x+1}{e^x-1}\phi_X'(x) &= -\frac{e^{2x}+1}{(e^x-1)^2} + \frac{(c_1-\lambda)e^{2x} - c_1}{(e^x-1)^2} \\ &= \frac{(c_1-\lambda-1)e^{2x} - c_1 - 1}{(e^x-1)^2}. \end{aligned}$$

Then $h_0 := \phi_X'$ solves

$$h_0'(x) + \frac{e^x+1}{e^x-1}h_0(x) = \frac{(c_1-\lambda-1)e^{2x} - c_1 - 1}{(e^x-1)^2}. \tag{4.4}$$

Note that $x \mapsto \frac{K}{4\sinh^2(x/2)}$ solves (4.4) with the right-hand side equal to 0, and $x \mapsto \frac{(c_1-\lambda-1)e^x+(c_1+1)e^{-x}}{4\sinh^2(x/2)}$ is a particular solution of (4.4). Therefore, the solution of (4.4) is

$$h(x) = \frac{(c_1-\lambda-1)e^x + (c_1+1)e^{-x} + K}{4\sinh^2(\frac{x}{2})}$$

for some constant K. This implies

$$\begin{aligned}
\phi'_X(x) &= \frac{(c_1-\lambda-1)e^x + (c_1+1)e^{-x} + K}{e^x + e^{-x} - 2} \\
&= \frac{(c_1-\lambda-1)e^{2x} + c_1 + 1 + Ke^x}{(e^x-1)^2} \\
&= c_1 + 1 + \frac{(2c_1-\lambda+K)e^x}{(e^x-1)^2} - \frac{(\lambda+2)e^x}{e^x-1}.
\end{aligned}$$

As a consequence, there exists a constant δ such that

$$\phi_X(x) = (c_1+1)x - \frac{(2c_1-\lambda+K)e^x}{e^x-1} - (\lambda+2)\log(e^x-1) + \delta.$$

Thus $p_X(x) = Ne^{(c_1+1)x}(e^x-1)^{-\lambda-2}\exp\left(-\frac{2c_1-\lambda+K}{e^x-1}\right)\mathbf{1}_{\{x>0\}}$. Recall that $a = -c_1$ and $b = \lambda+1$. With $c = 2c_1-\lambda+K$ one gets (1.21). More information on the constant N is obtained by observing that if we set $V' = f_1(X) = \frac{1}{e^X-1}$, then the density of V' is

$$f_{V'}(w) = N(w+1)^{-a}w^{a+b-1}\exp\{-cw\}\mathbf{1}_{\{w>0\}},$$

i.e. the law of V' is $K^{(2)}(a+b,-b,c)$ (cf Equation (1.18)).

We have

$$g'_1(u) = -\frac{1}{u(u+1)}.$$

Equation (2.3), together with (1.20) and (1.21), imply, for $u, v > 0$,

$$\begin{aligned}
p_{(U,V)}(u,v) &= p_X\left(\log\left[\frac{u+v+1}{u+v}\right]\right) p_Y\left(\log\left[\frac{(u+1)(u+v)}{u(u+v+1)}\right]\right) \\
&\quad \times \frac{1}{u(u+1)(u+v)(u+v+1)}.
\end{aligned}$$

Then we get that $p_{(U,V)}(u,v)$ is the product of a function of u and a function of v and this gives item 2. of Theorem 1.4.

$\square$

5 Appendix

5.1 Another proof of the result by Letac and Wesolowski under stronger assumptions

Proposition 5.1 *Consider two independent and positive random variables X and Y having positive and twice differentiable densities.*
If the random variables $U=(X+Y)^{-1}$ and $V=X^{-1}-(X+Y)^{-1}$ are independent then there exist $\mu>0$, $a>0$ and $b>0$ such that the law of X is GIG($-\mu,a,b$) and Y follows the gamma distribution $\gamma(\mu,\frac{b^2}{2})$.

Proof : We keep the notation introduced in Section 1. The densities of X and Y are respectively denoted by p_X and p_Y. Let $\phi_X=\log p_X$ and $\phi_Y=\log p_Y$. Theorem 2.1 with $f: x\mapsto 1/x$ asserts that the variables U and V are independent if and only if, for all $x,y>0$,

$$\phi_X''(x)+\frac{2}{x}\phi_X'(x)+\phi_Y''(y)\frac{1}{x^2}(x^2-(x+y)^2)-\frac{2}{x}\phi_Y'(y)+\frac{2}{x^2}=0. \tag{5.5}$$

Let us first compute the density of Y.
Equation (5.5) implies that, for fixed $x>0$, the function $\phi_Y'=h$ is a solution of the ordinary linear differential equation

$$\frac{2}{x}h(y)+h'(y)\frac{y^2+2xy}{x^2}=c(x),\quad y>0, \tag{5.6}$$

where $c(x)$ does not depend on y. The solutions of $h'(y)+\frac{2x}{y^2+2xy}h(y)=0$ are of the type

$$h(y)=k(x)\frac{y+2x}{y}.$$

Note that $y\mapsto -c(x)\frac{x^2}{y}$ is a particular solution of (5.6), consequently h solves (5.6) iff:

$$h(y)=-c(x)\frac{x^2}{y}+k(x)\frac{y+2x}{y},\quad y>0.$$

Recall that $\phi_Y'=h$, then we have $\phi_Y'(y)=\frac{1}{y}(-c(x)x^2+2xk(x)+yk(x))$. But ϕ_Y' is a function of y only. As a consequence, the expressions $-c(x)x^2+2xk(x)$ and $k(x)$ actually do not depend on x. We denote them respectively by δ and λ, so that

$$\phi_Y'(y)=\frac{1}{y}(\delta+\lambda y),\quad y>0,$$

which implies the existence of a constant $d\in\mathbb{R}$ such that

$$\phi_Y(y)=\delta\ln y+\lambda y+d,\quad y>0.$$

But $\phi_Y = \log p_Y$, so the density of Y is

$$p_Y(y) = \theta y^\delta e^{\lambda y}, \quad y > 0. \tag{5.7}$$

where $\theta = e^d$ is a positive constant. Since p_Y is, as a density, integrable, we have necessarily $\lambda < 0$ and $\delta > -1$ and defining

$$\mu := \delta + 1, \;\; b = \sqrt{-2\lambda}, \tag{5.8}$$

formula (5.7) shows that the law of Y is $\gamma(\mu, b^2/2)$.

We now compute the density of X.
We rewrite Equation (5.5) by replacing $\phi'_Y(y)$ with $\frac{1}{y}(\delta + \lambda y)$ and $\phi''_Y(y)$ with $\frac{-\delta}{y^2}$. This gives

$$\phi''_X(x) + \frac{2}{x}\phi'_X(x) + \frac{\delta+2}{x^2} - \frac{2\lambda}{x} = 0.$$

So ϕ'_X is a solution of the ordinary linear differential equation:

$$\kappa'(x) + \frac{2}{x}\kappa(x) + \frac{\delta+2}{x^2} - \frac{2\lambda}{x} = 0. \tag{5.9}$$

The solution of the corresponding homogeneous differential equation $\kappa'(x) + \frac{2}{x}\kappa(x) = 0$ is $\kappa(x) = \frac{\eta}{x^2}$, for some constant η $(x > 0)$. Note that $x \mapsto \frac{(-2-\delta)x+\lambda x^2}{x^2}$ solves (5.9), then κ solves (5.9) iff for some η, $\kappa(x) = \frac{(-2-\delta)x+\lambda x^2+\eta}{x^2}$. Since $\kappa = \phi'_X$ we have

$$\phi'_X(x) = \frac{-2-\delta}{x} + \lambda + \frac{\eta}{x^2}.$$

As a consequence, there exists a constant $\tau \in \mathbb{R}$ such that

$$\phi_X(x) = (-2-\delta)\ln x + \lambda x - \frac{\eta}{x} + \tau.$$

Since $\phi_X = \log p_X$, we have

$$p_X(x) = x^{-2-\delta}(\exp\tau)\exp(\lambda x - \frac{\eta}{x})$$

which proves that X follows the law $GIG(-\mu, a, b)$ with $\mu := \delta + 1$, $b = \sqrt{-2\lambda}$ (cf (5.8)) and $a = \sqrt{2\eta}$ (observe that the integrability of p_Y implies $\eta > 0$) .

$\square$

5.2 Proof of Lemma 3.7

We have

$$\begin{aligned}
F'(x+y)F(y) &= \sum_{n,m\geq 0} a_n a_m n(x+y)^{n-1}y^m \\
&= \sum_{n,m\geq 0} a_n a_m n y^m \left(\sum_{k=0}^{n-1} C_{n-1}^k x^k y^{n-1-k}\right) \\
&= \sum_{k\geq 0} x^k \sum_{m\geq 0, n\geq 1+k} n a_n a_m C_{n-1}^k y^{n+m-1-k}.
\end{aligned}$$

Setting $l = m + n - 1 - k$ for fixed m gives

$$F'(x+y)F(y) = \sum_{k\geq 0, l\geq 0} x^k y^l \sum_{m=0}^{l} (l-m+1+k) C_{l-m+k}^{k} a_{l-m+1+k} a_m. \tag{5.10}$$

$$\begin{aligned} F'(y)F(x+y) &= \sum_{n,m\geq 0} m a_n a_m (x+y)^n y^{m-1} \\ &= \sum_{n,m\geq 0} m a_n a_m y^m \left(\sum_{k=0}^{n} C_n^k x^k y^{n-k} \right) y^{m-1} \\ &= \sum_{k\geq 0} x^k \sum_{m\geq 0, n\geq k} m a_n a_m C_n^k y^{n+m-1-k}. \end{aligned}$$

We set again $l = m + n - 1 - k$ for fixed k :

$$F'(y)F(x+y) = \sum_{k\geq 0, l\geq 0} x^k y^l \left(\sum_{m=0}^{l+1} m C_{l-m+k+1}^{k} a_{l-m+1+k} a_m \right) \tag{5.11}$$

$$F'(y)F(x) = \sum_{k\geq 0, l\geq 0} a_k a_{l+1} (l+1) x^k y^l \tag{5.12}$$

$$F'(x)F(y) = \sum_{k\geq 0, l\geq 0} a_{k+1} a_l (k+1) x^k y^l \tag{5.13}$$

$$\begin{aligned} F'(0_+)F(x+y) &= a_1 \sum_{n\geq 0} a_n (x+y)^n \\ &= a_1 \sum_{n\geq 0} a_n \left(\sum_{k=0}^{n} C_n^k x^k y^{n-k} \right) \\ &= a_1 \sum_{k\geq 0} x^k \sum_{n\geq k} a_n C_n^k y^{n-k} \\ &= a_1 \sum_{k,l\geq 0} a_{l+k} C_{l+k}^{k} x^k y^l. \end{aligned} \tag{5.14}$$

$$F'(0_+)F(x) = a_1 \sum_{k\geq 0} a_k x^k. \tag{5.15}$$

Identifying the coefficient of $x^k y^l$ in (3.17) and using (5.10) to (5.15) we have, for $k \geq 0$ and $l \geq 0$:

$$\begin{aligned} \sum_{m=0}^{l} (l-m+1+k) C_{l-m+k}^{k} a_{l-m+1+k} a_m &= -(l+1) a_k a_{l+1} - (k+1) a_{k+1} a_l \\ &\quad + \sum_{m=0}^{l+1} m C_{l-m+k+1}^{k} a_{l-m+1+k} a_m \\ &\quad + a_1 a_{l+k} C_{l+k}^{k} - a_1 a_k 1_{l=0}. \end{aligned} \tag{5.16}$$

Note that if $l = 0$, both sides of (5.16) vanish, therefore we may suppose in the sequel that $l \geq 1$.
For $m = l + 1$ we have $mC^k_{l-m+k+1}a_{l-m+1+k}a_m = (l+1)a_k a_{l+1}$. Thus, Equation (5.16) reads

$$\sum_{m=0}^{l}(l-m+1+k)C^k_{l-m+k}a_{l-m+1+k}a_m = -(k+1)a_{k+1}a_l + \sum_{m=0}^{l} mC^k_{l-m+k+1}a_{l-m+1+k}a_m + a_1 a_{l+k}C^k_{l+k}. \tag{5.17}$$

But one finds by a calculation using the definition that

$$(l-m+1+k)C^k_{l-m+k} - mC^k_{l-m+1+k} = (l-2m+1)C^k_{l-m+1+k},$$

so that Equation (5.17) is equivalent to

$$\sum_{m=0}^{l}(l-2m+1)C^k_{l-m+1+k}a_{l-m+1+k}a_m = -(k+1)a_{k+1}a_l + a_1 a_{l+k}C^k_{l+k}. \tag{5.18}$$

For $m = l$ we have $(l-2m+1)C^k_{l-m+1+k}a_{l-m+1+k}a_m = (1-l)(k+1)a_{k+1}a_l$. Consequently Equation (5.18) may be written as follows:

$$\sum_{m=0}^{l-1}(l-2m+1)C^k_{l-m+1+k}a_{l-m+1+k}a_m - (l-1)(k+1)a_{k+1}a_l = -(k+1)a_{k+1}a_l + a_1 a_{l+k}C^l_{l+k}$$

which implies (3.14).
(3.15) and (3.16) follow by applying (3.14) to $l = 3$ and $l = 4$ respectively.

□

5.3 Proof of Theorem 1.14

Throughout this proof we write for simplicity f instead of f^*_δ. Starting with $f(x) = \log(e^x + \delta - 1) - \log(e^x - 1)$ we have

$$f'(x) = -\frac{\delta e^x}{(e^x + \delta - 1)(e^x - 1)} = \frac{-\delta}{e^x + \delta - 2 + (1-\delta)e^{-x}} \tag{5.19}$$

which implies $F(x) = 1/f'(x) = -\frac{1}{\delta}(e^x + \delta - 2 + (1-\delta)e^{-x})$ and $F'(0) = -1$. Therefore, again with $h = \phi'_Y$, Equation (3.8) is equivalent to

$$-\frac{h'(y)}{\delta}(e^y + \delta - 2 + (1-\delta)e^{-y}) - h(y) = \lambda,$$

whose solution is found to be, again by the method of variation of constants :

$$h(y) = c + \frac{\delta(c+\lambda)e^{-y}}{1 - e^{-y}}. \tag{5.20}$$

Thus, there exists a constant d' such that $\phi_Y(y) = cy + \delta(c+\lambda)\log(1-e^{-y}) + d'$ and this proves that $p_Y(y) = d(1-e^{-y})^{(c+\lambda)\delta} e^{cy}\,\mathbf{1}_{y>0}$. We set

$$a = -c,\ b = \delta(c+\lambda) + 1 \tag{5.21}$$

to obtain (1.36).

¿From (5.19) one has

$$f''(x) = \frac{\delta(e^x + (\delta-1)e^{-x})}{(e^x + \delta - 2 + (1-\delta)e^{-x})^2},$$

$$f'''(x) = \delta\frac{-e^{2x} + (\delta-2)e^x - 6\delta + 6 + (\delta-1)(2-\delta)e^{-x} - (\delta-1)^2e^{-2x}}{(e^x + \delta - 2 + (1-\delta)e^{-x})^3},$$

$$\frac{2(f''(x))^2 - f'''(x)f'(x)}{f'(x)^2} = \frac{e^x - (\delta-1)e^{-x}}{(e^x + \delta - 1)(1-e^{-x})},$$

$$-\phi_Y'(y)\frac{f''(x)}{f'(x)} - \phi_Y''(y)f'(x)\left(\frac{1}{f'(x)} - \frac{1}{f'(x+y)}\right) = -\frac{(a+b-1)e^x + a(\delta-1)e^{-x}}{(e^x + \delta - 1)(1-e^{-x})}.$$

Equation (2.1) is then equivalent to :

$$\phi_X''(x) + \frac{e^x + (\delta-1)e^{-x}}{(e^x + \delta - 1)(1-e^{-x})}\phi_X'(x) = \frac{(\delta-1)(1-a)e^{-x} - (a+b)e^x}{(e^x + \delta - 1)(1-e^{-x})}.$$

Solving this differential equation by similar calculations gives

$$\phi_X'(x) = \frac{K + (a-1)(\delta-1)e^{-x} - (a+b)e^x}{(e^x + \delta - 1)(1-e^{-x})}$$

where K is a constant. This can be written

$$\phi_X'(x) = A + \frac{Be^{-x}}{1 + (\delta-1)e^{-x}} + \frac{Ce^{-x}}{1-e^{-x}}$$

where

$$A = -a-b,\ B = \frac{(\delta-1)(K + 1 - a + (a+b)(\delta-1))}{\delta},\ C = \frac{K - a - b + (a-1)(\delta-1)}{\delta}.$$

This implies $\phi_X(x) = Ax - \frac{B}{\delta-1}\log(1 + (\delta-1)e^{-x}) + C\log(1-e^{-x}) + \kappa$ where κ is a constant. Thus one obtains

$$p_X(x) = De^{-(a+b)x}(1 + (\delta-1)e^{-x})^{-\lambda-b}(1-e^{-x})^{\lambda-1}\mathbf{1}_{x>0}$$

with $\lambda = C + 1 = \frac{B}{\delta-1} - b = \frac{K+1-b-2a+a\delta}{\delta}$. This proves (1.37).

To compute the density of (U, V) we note that $f = f^*_\delta$ is self-reciprocal. Then, by (2.3), the density of (U, V) is

$$\begin{aligned} p_{(U,V)}(u,v) &= p_X(f(u+v))\, p_Y(f(u) - f(u+v))\, f'(u+v) f'(u) \\ &= p_X\left(\log\left[\frac{e^{u+v}+\delta-1}{e^{u+v}-1}\right]\right) p_Y\left(\log\left[\frac{(e^u+\delta-1)(e^{u+v}-1)}{(e^u-1)(e^{u+v}+\delta-1)}\right]\right) \\ &\quad \times \frac{\delta^2}{(e^{u+v}+\delta-1)(1-e^{-u-v})(e^u+\delta-1)(1-e^{-u})}. \end{aligned}$$

Using the expressions of p_X and p_Y, one gets (1.38) and (1.39).

□

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